14th of April 2020

Mathematics exam (calculus section)

All exercices are independent and all answers must be justified (concisely). Grading instructions are indicative and subject to changes

Exercice 1 : Taylor expansion and limits, 6 points
Note $a, b > 0$. Compute :

$$\text{(A)} \lim_{x\to 6} \frac{\sin(\pi x)}{\ln(x-5)} \quad \text{(B)} \lim_{x\to b} \frac{x^a - b^a}{a^x - a^b} \quad \text{(C)} \lim_{x\to 0} \frac{e^x - \frac{1}{1-x}}{x^3}$$

Exercice 2 : Integration and probabilities (I), 2 points
Let $a > 0$. Consider the triangular function f_a defined on $\mathbb{R}$ by

$$f_a(x) = \begin{cases} Cx & \text{if } 0 \leq x \leq a \\ C(2a - x) & \text{if } a \leq x \leq 2a \\ 0 & \text{else} \end{cases}$$

A random variable X is triangular with parameter $a > 0$ if its distribution f_a. The k-th order moment of X, noted $\mathbb{E}[X^k]$ is given by $\mathbb{E}[X^k] = \int_{-\infty}^{+\infty} x^k f_a(x)dx$.

— Compute C so that f_a is a proper probability distribution, *i.e.* such that $\int_{-\infty}^{+\infty} f_a(x)dx = 1$.
— Compute the first and second order moment of X.

Exercice 3 : Integration and probabilities (II), 3 points
Let $a > 0$. Consider the function f_a defined on $\mathbb{R}$ by

$$f_a(x) = Ce^{-a|x|}$$

A random variable X is Laplace with parameter $a > 0$ if its probability density is f_a.

— Compute C such that $f_a(x)$ is a proper probability distribution.
— Compute the first and second order moment of X.

Exercice 4 : Partial derivatives, 3 points
Consider the function f defined from $\mathbb{R}^2$ to $\mathbb{R}$ as

$$f : (x, y) \mapsto x\cos(y) - y\cos(x)$$

Find the set $\mathcal{C}$ of all points (x, y) satisfying $\frac{\partial f^2}{\partial x \partial y}(x, y) = 0$. Draw the set $\mathcal{C}$ in the space $\mathbb{R}^2$.

Problem 1 : Geometry and optimisation, 5 points

Consider a rectangular sheet of metal with width $l = 3$ meteres and length $L = 5$ meters. You fold the sheet of metal along its length as follows :
The front and end side of the folded sheet are closed with appropriate lids to create an oddly shaped bucket. What angle θ maximizes the volume of the bucket ? What is the corresponding volume ?

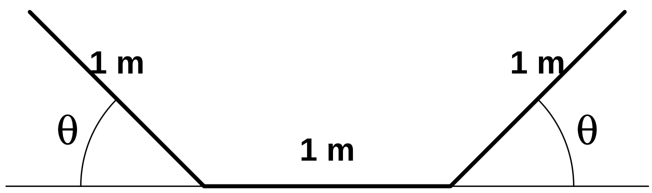

FIGURE 1 – Side view of the folded sheet of metal